THE DIFFEOMORPHISM TYPE OF MANIFOLDS WITH ALMOST MAXIMAL VOLUME

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Abstract. The smallest r so that a metric r–ball covers a metric space M is called the radius of M. The volume of a metric r-ball in the space form of constant curvature k is an upper bound for the volume of any Riemannian manifold with sectional curvature $\geq k$ and radius $\leq r$. We show that when such a manifold has volume almost equal to this upper bound, it is diffeomorphic to a sphere or a real projective space.

1. Introduction

Any closed Riemannian n–manifold M has a lower bound for its sectional curvature, $k \in \mathbb{R}$. This gives an upper bound for the volume of any metric ball $B(x,r) \subset M$,

$$\mathrm{vol}\, B(x,r) \leq \mathrm{vol}\, \mathcal{D}_k^n(r),$$

where $\mathcal{D}_k^n(r)$ is an r–ball in the n–dimensional, simply connected space form of constant curvature k. If rad M is the smallest number r such that a metric r–ball covers M, it follows that

$$\mathrm{vol}\, M \leq \mathrm{vol}\, \mathcal{D}_k^n(\mathrm{rad}\ M).$$

The invariant rad M is known as the *radius* of M and can alternatively be defined as

$$\mathrm{rad} M = \min_{p\in M} \max_{x\in M} \mathrm{dist}(p,x).$$

In the event that $\mathrm{vol}\, M$ is almost equal to $\mathrm{vol}\, \mathcal{D}_k^n(\mathrm{rad}\ M)$, we determine the diffeomorphism type of M.

Main Theorem. *Given $n \in \mathbb{N}, k \in \mathbb{R}$, and $r > 0$, there is an $\varepsilon > 0$ so that every closed Riemannian n–manifold M with*

$$\begin{aligned} \sec M &\geq k, \\ \mathrm{rad}\ M &\leq r, \text{ and} \\ \mathrm{vol}\ M &\geq \mathrm{vol}\, \mathcal{D}_k^n(r) - \varepsilon \end{aligned} \tag{1.0.1}$$

is diffeomorphic to S^n or $\mathbb{R}P^n$.

Grove and Petersen obtained the same result with *diffeomorphism* replaced by *homeomorphism* in [12]. They also showed that for any $\varepsilon > 0$ and $M = S^n$ or $\mathbb{R}P^n$ there are Riemannian metrics that satisfy (1.0.1) except when $k > 0$ and $r \in \left(\frac{1}{2}\frac{\pi}{\sqrt{k}}, \frac{\pi}{\sqrt{k}}\right)$.

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For $k>0$ and $r\in\left(\frac{1}{2}\frac{\pi}{\sqrt{k}},\frac{\pi}{\sqrt{k}}\right)$, Grove and Petersen also computed the optimal upper volume bound for the class of manifolds M with

$$\sec M\geq k \text{ and } \operatorname{rad} M\leq r. \tag{1.0.2}$$

It is strictly less than vol $\mathcal{D}_k^n(r)$ [12]. For $k>0$ and $r\in\left(\frac{1}{2}\frac{\pi}{\sqrt{k}},\frac{\pi}{\sqrt{k}}\right)$, manifolds satisfying (1.0.2) with almost maximal volume are already known to be diffeomorphic to spheres [14]. The main theorem in [24] gives the same result when $r=\frac{\pi}{\sqrt{k}}$.

For $k>0$ and $r=\frac{\pi}{\sqrt{k}}$, the maximal volume $\operatorname{vol}\mathcal{D}_1^n\left(\frac{\pi}{\sqrt{k}}\right)$ is realized by the n-sphere with constant curvature k. For $k>0$ and $r=\frac{\pi}{2\sqrt{k}}$, the maximal volume $\operatorname{vol}\mathcal{D}_1^n\left(\frac{\pi}{2\sqrt{k}}\right)$ is realized by $\mathbb{R}P^n$ with constant curvature k. Apart from these cases, there are no Riemannian manifolds M satisfying (1.0.2) and vol $M=$ vol $\mathcal{D}_k^n(r)$. Rather, the maximal volume is realized by one of two types of Alexandrov spaces. [12]

Example 1.1. ***(Crosscap)*** *The constant curvature k Crosscap, $C_{k,r}^n$, is the quotient of $\mathcal{D}_k^n(r)$ obtained by identifying antipodal points on the boundary. Thus $C_{k,r}^n$ is homeomorphic to $\mathbb{R}P^n$. There is a canonical metric on $C_{k,r}^n$ that makes this quotient map a submetry. The universal cover of $C_{k,r}^n$ is the double of $\mathcal{D}_k^n(r)$. If we write this double as $\mathbb{D}_k^n(r):=\mathcal{D}_k^n(r)^+\cup_{\partial\mathcal{D}_k^n(r)^\pm}\mathcal{D}_k^n(r)^-$, then the free involution*

$$A:\mathbb{D}_k^n(r)\longrightarrow\mathbb{D}_k^n(r)$$

that gives the covering map $\mathbb{D}_k^n(r)\longrightarrow C_{k,r}^n$ is

$$A:(x,+)\longmapsto(-x,-),$$

where the sign in the second entry indicates whether the point is in $\mathcal{D}_k^n(r)^+$ or $\mathcal{D}_k^n(r)^-$.

Example 1.2. ***(Purse)*** *Let $R:\mathcal{D}_k^n(r)\to\mathcal{D}_k^n(r)$ be reflection in a totally geodesic hyperplane H through the center of $\mathcal{D}_k^n(r)$. The Purse, $P_{k,r}^n$, is the quotient space*

$$\mathcal{D}_k^n(r)/\{v\sim R(v)\},\ \textit{provided } v\in\partial\mathcal{D}_k^n(r).$$

Alternatively we let $\{\mathcal{H}\mathcal{D}_k^n(r)\}^+\cup\{\mathcal{H}\mathcal{D}_k^n(r)\}^-=D_k^n(r)$ be the decomposition of $\mathcal{D}_k^n(r)$ into the two half disks on either side of H. Then $P_{k,r}^n$ is isometric to the double of $\{\mathcal{H}\mathcal{D}_k^n(r)\}^+$.

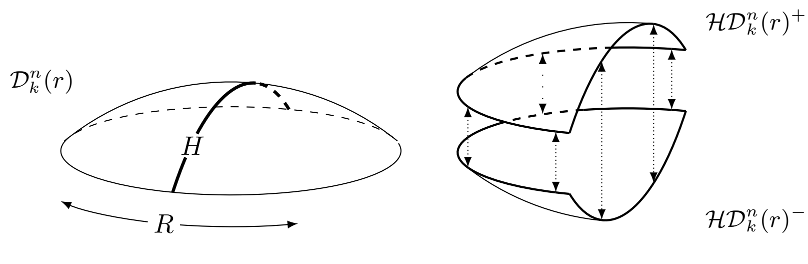

FIGURE 1. Two equivalent constructions of $P_{1,r}^2$

Let $\{M_i\}_{i=1}^\infty$ be a sequence of closed n-manifolds satisfying sec $M\geq k$ and rad $M\leq r$ and $\{\operatorname{vol}M_i\}$ converging to vol $\mathcal{D}_k^n(r)$ where $r\leq\frac{\pi}{2\sqrt{k}}$ if $k>0$. Grove and

Petersen showed that $\{M_i\}$ has a subsequence that converges to either $C^n_{k,r}$ or $P^n_{k,r}$ in the Gromov-Hausdorff topology [12]. The main theorem follows by combining this with the following *diffeomorphism stability theorems.*

Theorem 1.3. *Let $\{M_i\}_{i=1}^{\infty}$ be a sequence of closed Riemannian n–manifolds with $\sec M_i \geq k$ so that*

$$M_i \longrightarrow C^n_{k,r}$$

in the Gromov-Hausdorff topology. Then all but finitely many of the M_is are diffeomorphic to $\mathbb{R}P^n$.

Theorem 1.4. *Let $\{M_i\}_{i=1}^{\infty}$ be a sequence of closed Riemannian n–manifolds with $\sec M_i \geq k$ so that*

$$M_i \longrightarrow P^n_{k,r}$$

in the Gromov-Hausdorff topology. Then all but finitely many of the M_is are diffeomorphic to S^n.

Remark 1.5. *One can get Theorem 1.4 for the case $k = 1$ and $r > \operatorname{arccot}\left(\frac{1}{\sqrt{n-3}}\right)$ as a corollary of Theorem C in [15]. Theorem 1.3 when $k = 1$ and $r = \frac{\pi}{2}$ follows from the main theorem in [35] and the fact that $C^n_{1,\frac{\pi}{2}}$ is $\mathbb{R}P^n$ with constant curvature 1 . With minor modifications of our proof, the hypothesis $\sec M_i \geq k$ in Theorems 1.3 and 1.4 can replaced, except in one case, with an arbitrary uniform lower curvature bound. The exceptional case, is Theorem 1.3 in dimension 4, specifically in Proposition 5.3. For ease of notation, we have written all of the proofs for $\{M_i\}_{i=1}^{\infty}$ with $\sec M_i \geq k$ converging to $C^n_{k,r}$ or $P^n_{k,r}$.*

Section 2 introduces notations and conventions. Section 3 is review of necessary tools from Alexandrov geometry. Section 4 develops machinery and proves Theorem 1.3 in the case when $n \neq 4$. Theorem 1.3 in dimension 4 is proven in Section 5, and Theorem 1.4 is proven in Section 6.

Throughout the remainder of the paper, we assume without loss of generality, by rescaling if necessary, that $k = -1, 0$ or 1.

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2. Conventions and Notations

We assume a basic familiarity with Alexandrov spaces, including but not limited to [1]. Let X be an n–dimensional Alexandrov space and $x, p, y \in X$.

(1) We call minimal geodesics in X *segments.* We denote by px a segment in X with endpoints p and x.
(2) We let Σ_p and T_pX denote the space of directions and tangent cone at p, respectively.
(3) For $v \in T_pX$ we let γ_v be the segment whose initial direction is v.
(4) Following [28], $\Uparrow_x^p \subset \Sigma_x$ will denote the set of directions of segments from x to p, and $\uparrow_x^p \in \Uparrow_x^p$ denotes the direction of a single segment from x to p.
(5) We let $\sphericalangle(x, p, y)$ denote the angle of a hinge formed by px and py and $\tilde{\sphericalangle}(x, p, y)$ denote the corresponding comparison angle.
(6) Following [24], we let $\tau : \mathbb{R}^k \to \mathbb{R}_+$ be any function that satisfies

$$\lim_{x_1,\ldots,x_k \to 0} \tau(x_1, \ldots, x_k) = 0,$$

and abusing notation we let $\tau : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}$ be any function that satisfies

$$\lim_{x_1,\ldots,x_k \to 0} \tau\left(x_1,\ldots,x_k|y_1,\ldots,y_n\right)=0,$$

provided that $y_1,\ldots,y_n$ remain fixed.

When making an estimate with a function τ we implicitly assert the existence of such a function for which the estimate holds.

(7) We denote by $\mathbb{R}^{1,n}$ the Minkowski space $(\mathbb{R}^{n+1}, g)$, where g is the semi-Riemannian metric defined by

$$g=-dx_0^2+dx_1^2+\cdots+dx_n^2$$

for coordinates $(x_0,x_1,\cdots,x_n)$ on $\mathbb{R}^{n+1}$.

(8) We reserve $\{e_j\}_{j=0}^m$ for the standard orthonormal basis in both euclidean and Minkowski space.

(9) We use two isometric models for hyperbolic space,

$$H_+^n := \left\{ (x_0,x_1,\cdots,x_n)\in\mathbb{R}^{n+1} \middle| -(x_0)^2+(x_1)^2+\cdots+(x_n)^2=-1,\ x_0>0 \right\}$$

and

$$H_-^n := \left\{ (x_0,x_1,\cdots,x_n)\in\mathbb{R}^{n+1} \middle| -(x_0)^2+(x_1)^2+\cdots+(x_n)^2=-1,\ x_0<0 \right\}.$$

(10) We obtain explicit double disks, $\mathbb{D}_k^n(r) := \mathcal{D}_k^n(r)^+ \cup_{\partial\mathcal{D}_k^n(r)^\pm} \mathcal{D}_k^n(r)^-$, by viewing $\mathcal{D}_k^n(r)^+$ and $\mathcal{D}_k^n(r)^-$ explicitly as

$$\mathcal{D}_k^n(r)^+ := \left[\begin{array}{cc} \left\{ z\in H_+^n\subset\mathbb{R}^{1,n} \middle| \operatorname{dist}_{H_+^n}(e_0,z)\le r \right\} & \text{if } k=-1 \\ \left\{ z\in\{e_0\}\times\mathbb{R}^n\subset\mathbb{R}^{n+1} \middle| \operatorname{dist}_{\mathbb{R}^{n+1}}(e_0,z)\le r \right\} & \text{if } k=0 \\ \left\{ z\in S^n\subset\mathbb{R}^{n+1} \middle| \operatorname{dist}_{S^n}(e_0,z)\le r \right\} & \text{if } k=1, \end{array}\right.$$

and

$$\mathcal{D}_k^n(r)^- := \left[\begin{array}{cc} \left\{ z\in H_-^n\subset\mathbb{R}^{1,n} \middle| \operatorname{dist}_{H_-^n}(-e_0,z)\le r \right\} & \text{if } k=-1 \\ \left\{ z\in\{-e_0\}\times\mathbb{R}^n\subset\mathbb{R}^{n+1} \middle| \operatorname{dist}_{\mathbb{R}^{n+1}}(-e_0,z)\le r \right\} & \text{if } k=0 \\ \left\{ z\in S^n\subset\mathbb{R}^{n+1} \middle| \operatorname{dist}_{S^n}(-e_0,z)\le r \right\} & \text{if } k=1. \end{array}\right.$$

Since $r<\frac{\pi}{2}$ when $k=1$, $\mathcal{D}_k^n(r)^+$ and $\mathcal{D}_k^n(r)^-$ are disjoint in all three cases.

3. Basic Tools From Alexandrov Geometry

The notion of strainers [1] in an Alexandrov space forms the core of the calculus arguments used to prove our main theorem. In this section, we review this notion and its relevant consequences. In some sense the idea can be traced back to [24], and some of the ideas that we review first appeared in other sources such as [34] and [36].

Definition 3.1. *Let X be an Alexandrov space. A point $x\in X$ is said to be (n,δ,r)–strained by the strainer $\{(a_i,b_i)\}_{i=1}^n \subset X\times X$ provided that for all $i\neq j$ we have*

$$\begin{array}{c} \tilde{\sphericalangle}(a_i,x,b_j)>\frac{\pi}{2}-\delta,\quad \tilde{\sphericalangle}(a_i,x,b_i)>\pi-\delta, \\ \tilde{\sphericalangle}(a_i,x,a_j)>\frac{\pi}{2}-\delta,\quad \tilde{\sphericalangle}(b_i,x,b_j)>\frac{\pi}{2}-\delta,\ \text{and} \\ \min_{i=1,\ldots,n}\{\operatorname{dist}(\{a_i,b_i\},x)\}>r. \end{array}$$

We say a metric ball $B\subset X$ is an (n,δ,r)–strained neighborhood with strainer $\{a_i,b_i\}_{i=1}^n$ provided every point $x\in B$ is (n,δ,r)–strained by $\{a_i,b_i\}_{i=1}^n$.

The following is observed in [36].

Proposition 3.2. *Let X be a compact n-dimensional Alexandrov space. Then the following are equivalent.*

1: *There is a (sufficiently small) $\eta > 0$ so that for every $p \in X$*
$$\mathrm{dist}_{G-H}\left(\Sigma_p, S^{n-1}\right) < \eta.$$

2: *There is a (sufficiently small) $\delta > 0$ and an $r > 0$ such that X is covered by finitely many (n, δ, r)–strained neighborhoods.*

Theorem 3.3. *([1] Theorem 9.4) Let X be an n–dimensional Alexandrov space with curvature bounded from below. Let $p \in X$ be (n, δ, r)–strained by $\{(a_i, b_i)\}_{i=1}^n$. Provided δ is small enough, there is a $\rho > 0$ such that the map $f : B(p, \rho) \to \mathbb{R}^n$ defined by*
$$f(x) = \left(\mathrm{dist}\left(a_1, x\right), \mathrm{dist}\left(a_2, x\right), \ldots, \mathrm{dist}\left(a_n, x\right)\right)$$
is a bi-Lipschitz embedding with Lipschitz constants in $\left(1 - \tau\left(\delta, \rho\right), 1 + \tau\left(\delta, \rho\right)\right)$.

If every point in X is (n, δ, r)–strained, we can equip X with a C^1–differentiable structure defined by Otsu and Shioya in [25]. The charts will be smoothings of the map from the theorem above and are defined as follows: Let $x \in X$ and choose $\sigma > 0$ so that $B(x, \sigma)$ is (n, δ, r)–strained by $\{a_i, b_i\}_{i=1}^n$. Define $d_{i,x}^\eta : B(x, \sigma) \to \mathbb{R}$ by
$$d_{i,x}^\eta(y) = \frac{1}{\mathrm{vol}(B(a_i, \eta))} \int_{z \in B(a_i, \eta)} \mathrm{dist}(y, z).$$
Then $\varphi_x^\eta : B(x, \sigma) \to \mathbb{R}^n$ is defined by
$$\varphi_x^\eta(y) = (d_{1,x}^\eta(y), \ldots, d_{n,x}^\eta(y)). \tag{3.3.1}$$

If B is (n, δ, r)–strained by $\{a_i, b_i\}_{i=1}^n$, any choice of $2n$–directions $\left\{\left(\uparrow_x^{a_i}, \uparrow_x^{b_i}\right)\right\}_{i=1}^n$ where $x \in B$ will be called a set of straining directions for Σ_x. As in, [1, 36], we say an Alexandrov space Σ with $\mathrm{curv}\, \Sigma \geq 1$ is globally (m, δ)-strained by pairs of subsets $\{A_i, B_i\}_{i=1}^m$ provided
$$\begin{array}{ll} |\mathrm{dist}(a_i, b_j) - \frac{\pi}{2}| < \delta, & \mathrm{dist}(a_i, b_i) > \pi - \delta, \\ |\mathrm{dist}(a_i, a_j) - \frac{\pi}{2}| < \delta, & |\mathrm{dist}(b_i, b_j) - \frac{\pi}{2}| < \delta \end{array}$$
for all $a_i \in A_i$, $b_i \in B_i$ and $i \neq j$.

Theorem 3.4. *([1] Theorem 9.5, cf also [24] Section 3) Let Σ be an $(n-1)$–dimensional Alexandrov space with curvature ≥ 1. Suppose Σ is globally strained by $\{A_i, B_i\}$. There is a map $\tilde{\Psi} : \mathbb{R}^n \longrightarrow S^{n-1}$ so that $\Psi : \Sigma \to S^{n-1}$ defined by*
$$\Psi(x) = \tilde{\Psi} \circ \left(\mathrm{dist}\left(A_1, x\right), \mathrm{dist}\left(A_2, x\right), \ldots, \mathrm{dist}\left(A_n, x\right)\right)$$
is a bi-Lipschitz homeomorphisms with Lipshitz constants in $\left(1 - \tau\left(\delta\right), 1 + \tau\left(\delta\right)\right)$.

Remark 3.5. *The description of $\tilde{\Psi} : \mathbb{R}^n \longrightarrow S^{n-1}$ in [1] is explicit but is geometric rather than via a formula. Combining the proof in [1] with a limiting argument, one can see that the map Ψ can be given by*
$$\Psi(x) = \left(\sum \cos^2\left(\mathrm{dist}\left(A_i, x\right)\right)\right)^{-1/2} \left(\cos\left(\mathrm{dist}\left(A_1, x\right)\right), \ldots, \cos\left(\mathrm{dist}\left(A_n, x\right)\right)\right).$$
In particular, the differentials of $\varphi_x^\eta : B(x, \sigma) \subset X \longrightarrow \varphi(B(x, \sigma))$ are almost isometries.

Next we state a powerful lemma showing that for an (n,δ,r) strained neighborhood, angle and comparison angle almost coincide for geodesic hinges with one side in this neighborhood and the other reaching a strainer.

Lemma 3.6. *([1] Lemma 5.6) Let $B\subset X$ be $(1,\delta,r)$–strained by (y_1,y_2). For any $x,z\in B$*

$$\left|\tilde{\sphericalangle}(y_1,x,z)+\tilde{\sphericalangle}(y_2,x,z)-\pi\right|<\tau\left(\delta,\operatorname{dist}(x,z)\,|r\right)$$

In particular, for $i=1,2$,

$$\left|\sphericalangle(y_i,x,z)-\tilde{\sphericalangle}(y_i,x,z)\right|<\tau\left(\delta,\operatorname{dist}(x,z)\,|r\right).$$

Corollary 3.7. *Let $B\subset X$ be $(1,\delta,r)$–strained by (a,b). Let $\{X^\alpha\}_{\alpha=1}^\infty$ be a sequence of Alexandrov spaces with $\operatorname{curv}X^\alpha\geq k$ such that $X^\alpha\longrightarrow X$. For $x,z\in B$, suppose that $a^\alpha,b^\alpha,x^\alpha,z^\alpha\in X^\alpha$ converge to $a,b,x,$ and z respectively. Then*

$$\left|\sphericalangle(a^\alpha,x^\alpha,z^\alpha)-\sphericalangle(a,x,z)\right|<\tau\left(\delta,\operatorname{dist}(x,z),\tau\left(1/\alpha|\operatorname{dist}(x,z)\right)\ |\ r\right).$$

Proof. The convergence $X^\alpha\longrightarrow X$ implies that we have convergence of the corresponding comparison angles. The result follows from the previous lemma. □

Lemma 3.8. *Let $B\subset X$ be (n,δ,r)–strained by $\{(a_i,b_i)\}_{i=1}^n$. Let $\{X^\alpha\}_{\alpha=1}^\infty$ have $\operatorname{curv}X^\alpha\geq k$ and suppose that $X_\alpha\longrightarrow X$. Let $\{(\gamma_{1,\alpha},\gamma_{2,\alpha})\}_{\alpha=1}^\infty$ be a sequence of geodesic hinges in the X^α that converge to a geodesic hinge (γ_1,γ_2) with vertex in B. Then*

$$\left|\sphericalangle\left(\gamma_{1,\alpha}'(0),\gamma_{2,\alpha}'(0)\right)-\sphericalangle\left(\gamma_1'(0),\gamma_2'(0)\right)\right|<\tau\left(\delta,\tau\left(1/\alpha|\operatorname{len}(\gamma_1),\operatorname{len}(\gamma_2)\right)\ |\ r\right).$$

Remark 3.9. *Note that without the strainer, $\liminf_{\alpha\to\infty}\sphericalangle\left(\gamma_{1,\alpha}'(0),\gamma_{2,\alpha}'(0)\right)\geq\sphericalangle\left(\gamma_1'(0),\gamma_2'(0)\right)$ [11], [1].*

Proof. Apply the previous corollary with $x^\alpha=\gamma_{1,\alpha}(0)$, $z^\alpha=\gamma_{1,\alpha}(\varepsilon)$, $x^\alpha\to x$, and $z^\alpha\to z$ to conclude

$$\left|\sphericalangle(\Uparrow_{x^\alpha}^{a_i^\alpha},\gamma_{1,\alpha}'(0))-\sphericalangle(\Uparrow_x^{a_i},\gamma_1'(0))\right|<\tau\left(\delta,\operatorname{dist}(x,z),\tau\left(1/\alpha|\operatorname{dist}(x,z)\right)\ |\ r\right).$$

Similar reasoning with $x^\alpha=\gamma_{2,\alpha}(0)$, $z^\alpha=\gamma_{2,\alpha}(\varepsilon)$, $x=\lim_{\alpha\to\infty}x^\alpha$, and $z=\lim_{\alpha\to\infty}z^\alpha$ gives

$$\left|\sphericalangle(\Uparrow_{x^\alpha}^{a_i^\alpha},\gamma_{2,\alpha}'(0))-\sphericalangle(\Uparrow_x^{a_i},\gamma_2'(0))\right|<\tau\left(\delta,\operatorname{dist}(x,z),\tau\left(1/\alpha|\operatorname{dist}(x,z)\right)\ |\ r\right).$$

Since $\operatorname{dist}(x,z)$ may be as small as we please, the result then follows from Theorem 3.4. □

Lemma 3.10. *([36] Lemma 1.8.2) Let $\{(a_i,b_i)\}_{i=1}^n$ be an (n,δ,r)–strainer for $B\subset X$. For any $x\in B$ and $\mu>0$, let Σ_x^μ be the set of directions $v\in\Sigma_x$ so that $\gamma_v|_{[0,\mu]}$ is a segment. For any sufficiently small $\mu>0$, Σ_x^μ is $\tau(\delta,\mu)$–dense in Σ_x.*

Corollary 3.11. *Suppose $X^\alpha\longrightarrow X$, $\{(a_i,b_i)\}_{i=1}^n$ is an (n,δ,r)–strainer for $B\subset X$, and (n,δ,r)–strainers $\{(a_i^\alpha,b_i^\alpha)\}_{i=1}^n$ for $B^\alpha\subset X^\alpha$ satisfy*

$$\left(\{(a_i^\alpha,b_i^\alpha)\}_{i=1}^n,B^\alpha\right)\longrightarrow\left(\{(a_i,b_i)\}_{i=1}^n,B\right).$$

For any fixed $\mu>0$ and any sequence of directions $\{v^\alpha\}_{a=1}^\infty\subset\Sigma_{x^\alpha}$ with $x^\alpha\in B^\alpha$, there is a sequence $\{w^\alpha\}_{a=1}^\infty\subset\Sigma_{x^\alpha}^\mu$ with

$$\sphericalangle(w^\alpha,v^\alpha)<\tau(\delta,\mu)$$

so that a subsequence of $\{\gamma_{w^\alpha}\}_{\alpha=1}^\infty$ converges to a geodesic $\gamma:[0,\mu]\longrightarrow X$.

From Arzela-Ascoli and Hopf-Rinow, we conclude

Proposition 3.12. *Let X be an Alexandrov space and $p,q \in X$. For any $\varepsilon > 0$, there is a $\delta > 0$ so that for all $x \in B(p,\delta)$ and all $y \in B(q,\delta)$ and any segment xy, there is a segment pq so that*

$$\operatorname{dist}(xy, pq) < \varepsilon.$$

We end this section by showing that convergence to a compact Alexandrov space X without collapse implies the convergence of the corresponding universal covers, provided $|\pi_1(X)| < \infty$. For our purposes, when $X = C^n_{k,r}$, it would be enough to use [31] or [7].

The key tools are Perelman's Stability and Local Structure Theorems and the notion of first systole, which is the length of the shortest closed non-contractible curve. Perelman's proof of the Local Structure Theorem can be found in [27], this result is also a corollary to his Stability Theorem, whose proof is published in [16].

Theorem 3.13. *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of n–dimensional Alexandrov spaces with a uniform lower curvature bound converging to a compact, n–dimensional Alexandrov space X. If the fundamental group of X is finite, then*

- **1:** *A subsequence of the universal covers, $\{\tilde{X}_i\}_{i=1}^{\infty}$, of $\{X_i\}_{i=1}^{\infty}$converges to the universal cover, $\tilde{X}$, of X.*
- **2:** *A subsequence of the deck action by $\pi_1(X_i)$ on $\{\tilde{X}_i\}_{i=1}^{\infty}$ converges to the deck action of $\pi_1(X)$ on $\tilde{X}$.*

Proof. In [27], Perelman shows X is locally contractible. Let $\{U_j\}_{j=1}^{n}$ be an open cover of X by contractible sets and let μ be a Lebesgue number of this cover. By Perelman's Stability Theorem, there are $\tau\left(\frac{1}{i}\right)$–Hausdorff approximations

$$h_i : X \longrightarrow X_i$$

that are also homeomorphisms. Therefore, if i is sufficiently large, $\{h_i(U_j)\}_{j=1}^{n}$ is an open cover for X_i by contractible sets with Lebesgue number $\mu/2$. It follows that the first systoles of the X_is are uniformly bounded from below by μ. Since the minimal displacement of the deck transformations by $\pi_1(X_i)$ on $\tilde{X}_i \longrightarrow X_i$ is equal to the first systole of X_i, this displacement is also uniformly bounded from below by μ. By precompactness, a subsequence of $\{\tilde{X}_i\}$ converges to a length space Y. From Proposition 3.6 of [7], a subsequence of the actions $\left(\tilde{X}_i, \pi_1(X_i)\right)$ converges to an isometric action by some group G on Y. By Theorem 2.1 in [6], $X = Y/G$. Since the displacements of the (nontrivial) deck transformations by $\pi_1(X_i)$ on $\tilde{X}_i \longrightarrow X_i$ are uniformly bounded from below, the action by G on Y is properly discontinuous. Hence $Y \longrightarrow Y/G = X$ is a covering space of X. By the Stability Theorem, Y is simply connected, so Y is the universal cover of X. $\square$

Remark 3.14. *When the X_i are Riemannian manifolds, one can get the uniform lower bound for the systoles of the X_is from the generalized Butterfly Lemma in* [10]*. The same argument also works in the Alexandrov case but requires Perelman's critical point theory, and hence is no simpler than what we presented above.*

Lens spaces show that without the noncollapsing hypothesis this result is false even in constant curvature.

4. Cross Cap Stability

The main step to prove Theorem 1.3 is the following.

Theorem 4.1. *Let $\{M^\alpha\}_{\alpha=1}^\infty$ be a sequence of closed Riemannian n–manifolds with* $\sec M^\alpha \geq k$ *so that*

$$M^\alpha \longrightarrow C^n_{k,r}$$

in the Gromov-Hausdorff topology. Let $\tilde{M}^\alpha$ be the universal cover of M^α. Then for all but finitely many α, there is a C^1 embedding

$$\tilde{M}^\alpha \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\}$$

that is equivariant with respect to the deck transformations of $\tilde{M}^\alpha \longrightarrow M^\alpha$ and the Z_2–action on R^{n+1} generated by $-id$.

Two and three manifolds have unique differential structures up to diffeomorphism; so in dimensions two and three Theorems 1.3 and 4.1 follow from the main result of [12]. We give the proof in dimension 4 in section 6. Until then, we assume that $n \geq 5$.

Proof of Theorem 1.3 modulo Theorem 4.1. By Perelman's Stability Theorem all but finitely many $\{\tilde{M}^\alpha\}_{\alpha=1}^\infty$ are homeomorphic to S^n (cf [12]). Combining this with Theorem 4.1 and Brown's Theorem 9.7 in [22] gives an H–cobordism between the embedded image of $\tilde{M}^\alpha \subset \mathbb{R}^{n+1}$ and the standard S^n. Modding out by $\mathbb{Z}_2$, we see that M^α and $\mathbb{R}P^n$ are H–cobordant. Since the Whitehead group of $\mathbb{Z}_2$ is trivial ([18], [23], p. 373), any H–cobordism between M_α and $\mathbb{R}P^n$ is an S–cobordism and hence a product, which completes the proof. [2, 21, 32] $\square$

The proof of Theorem 1.3 does not exploit any a priori differential structure on the Crosscap. Instead we exploit a model embedding of the double disk

$$\mathbb{D}^n_k(r) \hookrightarrow \mathbb{R}^{n+1},$$

whose restriction to either half, $\mathcal{D}^n_k(r)^+$ or $\mathcal{D}^n_k(r)^-$, is the identity on the last n–coordinates. By describing the identity $\mathcal{D}^n_k(r) \longrightarrow \mathcal{D}^n_k(r)$ in terms of distance functions, we then argue that this embedding can be lifted to all but finitely many of a sequence $\{M^\alpha\}$ converging to $\mathbb{D}^n_k(r)$.

The Model Embedding. Let $A : \mathbb{D}^n_k(r) \to \mathbb{D}^n_k(r)$ be the free involution mentioned in Example 1.1. For $z \in \mathbb{D}^n_k(r)$, we define $f_z : \mathbb{D}^n_k(r) \to \mathbb{R}$ by

$$f_z(x) = h_k \circ \operatorname{dist}(A(z), x) - h_k \circ \operatorname{dist}(z, x) \tag{4.1.1}$$

where $h_k : \mathbb{R} \to \mathbb{R}$ is defined as

$$h_k(x) = \begin{cases} \frac{1}{2\sinh r}\cosh(x) & \text{if } k = -1 \\ \frac{x^2}{4r} & \text{if } k = 0 \\ \frac{1}{2\sin r}\cos(x) & \text{if } k = 1. \end{cases}$$

Recall that we view $\mathcal{D}^n_k(r)^\pm$ as metric r-balls centered at $p_0 = e_0$ and $A(p_0) = -e_0$ in either $H^n_\pm$, $\{\pm e_0\} \times \mathbb{R}^n$, or S^n. For $i = 1, 2, \ldots, n$ we set

$$p_i := \begin{cases} \cosh(r)e_0 + \sinh(r)e_i & \text{if } k = -1 \\ e_0 + re_i & \text{if } k = 0 \\ \cos(r)e_0 - \sin(r)e_i & \text{if } k = 1. \end{cases} \tag{4.1.2}$$

The functions $\{f_i\}_{i=1}^n := \{f_{p_i}\}_{i=1}^n$ are then restrictions of the last n–coordinate functions of $\mathbb{R}^{n+1}$ to $\mathcal{D}_k^n(r)^{\pm}$. We set $f_0 := f_{p_0}$. In contrast to $f_1, \ldots, f_n$, our f_0 is not a coordinate function. On the other hand its gradient is well defined everywhere on $\mathbb{D}_k^n(r) \setminus \{p_0, A(p_0)\}$, even on $\partial\mathcal{D}_k^n(r)^+ = \partial\mathcal{D}_k^n(r)^-$ where it is normal to $\partial\mathcal{D}_k^n(r)^+ = \partial\mathcal{D}_k^n(r)^-$.

Define $\Phi : \mathbb{D}_k^n(r) \to \mathbb{R}^{n+1}$, by

$$\Phi = (f_0, f_1, f_2, \cdots, f_n),$$

and observe that

Proposition 4.2. *Φ is a continuous, $\mathbb{Z}_2$–equivariant embedding.*

Proof. Write $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ and let $\pi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be projection. Since $f_1, f_2, \cdots, f_n$ are coordinate functions, the restrictions

$$\pi \circ \Phi|_{\mathcal{D}_k^n(r)^{\pm}} : \mathcal{D}_k^n(r)^{\pm} \longrightarrow \mathbb{R}^n$$

are both the identity. From this and the definition of f_0, we conclude that Φ is one–to–one. Since $\mathbb{D}_k^n(r)$ is compact, it follows that Φ is an embedding. The $\mathbb{Z}_2$–equivariance is immediate from definition 4.1.1. $\square$

Lifting the Model Embedding. To start the proof of Theorem 4.1 let $\{M^\alpha\}_{\alpha=1}^\infty$ be a sequence of closed Riemannian n–manifolds with sec $M^\alpha \geq k$ so that

$$M^\alpha \longrightarrow C_{k,r}^n,$$

and we let $\{\tilde{M}^\alpha\}_{\alpha=1}^\infty$ denote the corresponding sequence of universal covers. From Theorem 3.13, a subsequence of $\{\tilde{M}^\alpha\}_{\alpha=1}^\infty$ together with the deck transformations $\tilde{M}^\alpha \longrightarrow M^\alpha$ converge to $(\mathbb{D}_k^n(r), A)$. For all but finitely many α, $\pi_1(M^\alpha)$ is isomorphic to $\mathbb{Z}_2$. We abuse notation and call the nontrivial deck transformation of $\tilde{M}^\alpha \longrightarrow M^\alpha$, A.

First we extend definition 4.1.1 by letting $f_z^\alpha : \tilde{M}^\alpha \to \mathbb{R}$ be defined by

$$f_z^\alpha(x) = h_k \circ \operatorname{dist}(A(z), x) - h_k \circ \operatorname{dist}(z, x). \tag{4.2.1}$$

Let $p_i^\alpha \in \tilde{M}^\alpha$ converge to $p_i \in \mathbb{D}_k^n(r)$, and for some $d > 0$ define $f_{i,d}^\alpha : \tilde{M}^\alpha \to \mathbb{R}$ by

$$f_{i,d}^\alpha(x) = \frac{1}{\operatorname{vol} B(p_i^\alpha, d)} \int_{q^\alpha \in B(p_i^\alpha, d)} f_{q^\alpha}^\alpha(x). \tag{4.2.2}$$

Differentiation under the integral gives

Proposition 4.3. *The $f_{i,d}^\alpha$ are C^1 and $\left|\nabla f_{i,d}^\alpha\right| \leq 2$.*

We now define $\Phi_d^\alpha : \tilde{M}^\alpha \to \mathbb{R}^{n+1}$ by

$$\Phi_d^\alpha = \left(f_{0,d}^\alpha, f_{1,d}^\alpha, f_{2,d}^\alpha, \cdots, f_{n,d}^\alpha\right).$$

As $\alpha \to \infty$ and $d \to 0$, Φ_d^α converges to Φ in the Gromov–Hausdorff sense. Since Φ is an embedding it follows that Φ_d^α is one–to–one in the large. More precisely,

Proposition 4.4. *For any $\nu > 0$, if α is sufficiently large and d is sufficiently small, then*

$$\Phi_d^\alpha(x) \neq \Phi_d^\alpha(y),$$

provided $\operatorname{dist}(x, y) > \nu$.

Since the $\mathbb{Z}_2$-equivariance of Φ_d^α immediately follows from definition 4.2.2, all that remains to prove Theorem 4.1 is the following proposition:

Proposition 4.5. *There is a $\rho > 0$ so that Φ_d^α is one to one on all ρ–balls, provided that α is sufficiently large and d is sufficiently small.*

This is a consequence of Key Lemma 4.7 (stated below), whose statement and proof occupy the remainder of this section.

Uniform Immersion. The proof of the Inverse Function Theorem in [29] gives

Theorem 4.6. *(Quantitative Immersion Theorem) Let*

$$\mathbb{R}^n_{\hat{i}} := \{(x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{n+1})\} \subset \mathbb{R}^{n+1}$$

and let

$$P_{\hat{i}} : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n_{\hat{i}}$$

be orthogonal projection.

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ be a C^1 map so that for some $a \in \mathbb{R}^n$, $\lambda > 0$, and $\rho > 0$, there is an $i \in \{1, \ldots, n+1\}$ so that

$$\left| d\left(P_{\hat{i}} \circ F\right)_a (v) \right| \geq \lambda \left| v \right|$$

and

$$\left| d\left(P_{\hat{i}} \circ F\right)_a (v) - d\left(P_{\hat{i}} \circ F\right)_x (v) \right| < \frac{\lambda}{2} \left| v \right|$$

for all $x \in B(a, \rho)$ and $v \in \mathbb{R}^n$, then $\left(P_{\hat{i}} \circ F\right)|_{B(a,\rho)}$ is a one–to–one, open map.

We note that every space of directions to $\mathbb{D}^n_k(r)$ is isometric to S^{n-1}. By proposition 3.2, there are $r, \delta > 0$ so that every point in the double disk has a neighborhood B that is (n, δ, r)–strained. If $B \subset \mathbb{D}^n_k(r)$ is (n, δ, r)–strained by $\{a_i, b_i\}_{i=1}^n$, by continuity of comparison angles, we may assume there are sets $B^\alpha \subset \tilde{M}^\alpha$ (n, δ, r)–strained by $\{a_i^\alpha, b_i^\alpha\}_{i=1}^n$ such that

$$\left(\{(a_i^\alpha, b_i^\alpha)\}_{i=1}^n, B^\alpha \right) \longrightarrow \left(\{(a_i, b_i)\}_{i=1}^n, B \right).$$

Given $x^\alpha \in B^\alpha$, we let $\varphi^\eta_{x^\alpha}$ be as in 3.3.1.

To prove Proposition 4.5 it suffices to prove the following.

Key Lemma 4.7. *There is a $\lambda > 0$ and $\rho > 0$ so that for all $x^\alpha \in \tilde{M}^\alpha$ there is an $i_{x^\alpha} \in \{0, 1, \ldots, n\}$ such that the function $F := \Phi_d^\alpha \circ \left(\varphi^\eta_{x^\alpha}\right)^{-1}$ satisfies*

(1)
$$\left| d(P_{\hat{i}_{x^\alpha}} \circ F)_{\varphi^\eta_{x^\alpha}(x^\alpha)} (v) \right| > \lambda \left| v \right|$$

and

(2)
$$\left| d\left(P_{\hat{i}_{x^\alpha}} \circ F\right)_{\varphi^\eta_{x^\alpha}(y)} (v) - d\left(P_{\hat{i}_{x^\alpha}} \circ F\right)_{\varphi^\eta_{x^\alpha}(x^\alpha)} (v) \right| < \frac{\lambda}{2} \left| v \right|$$

for all $y \in B(x^\alpha, \rho)$ and $v \in \mathbb{R}^n$, provided that α is sufficiently large and d and η are sufficiently small.

We show in the next subsection that part 1 of Key Lemma 4.7 holds, and in the following subsection we show that part 2 holds.

Lower bound on the differential. We begin by illustrating that, in a sense, the first part of the key lemma holds for the model embedding.

Lemma 4.8. *There is a $\lambda > 0$ so that for all $v \in T\mathbb{D}_k^n(r)$ there is a $j(v) \in \{0, 1, \ldots, n\}$ so that*

$$\left|D_v f_{j(v)}\right| > \lambda |v|.$$

Proof. Recall that the double disk $\mathbb{D}_k^n(r)$ is the union of two copies of $\mathcal{D}_k^n(r)$ that we call $\mathcal{D}_k^n(r)^+$ and $\mathcal{D}_k^n(r)^-$—glued along their common boundary—that throughout this section we call $\mathcal{S} := \partial \mathcal{D}_k^n(r)^\pm$.

If $x \in \mathbb{D}_k^n(r) \setminus \mathcal{S}$, then for $i \neq 0$, ∇f_i is unambiguously defined; moreover,

$$\{\nabla f_i(x)\}_{i=1}^n$$

is an orthonormal basis. Thus the lemma certainly holds on $\mathbb{D}_k^n(r) \setminus \mathcal{S}$.

For $x \in \mathcal{S}$ and $i \in \{1, \ldots, n\}$, we can think of the gradient of f_i as multivalued. More precisely, for $x \in \mathcal{S}$, we view

$$\mathcal{S} \subset \mathcal{D}_k^n(r)^\pm \subset \begin{cases} H_\pm^n & \text{if } k = -1 \\ \{\pm e_0\} \times \mathbb{R}^n & \text{if } k = 0 \\ S^n & \text{if } k = 1 \end{cases}$$

and define $\nabla f_i^\pm$ to be the gradient at x of the coordinate function that extends f_i to either $H_\pm^n$, $\{\pm e_0\} \times \mathbb{R}^n$, or S^n.

From definition 4.1.1, for any $v \in T_x\mathbb{D}_k^n(r)$

$$D_v f_i = \begin{cases} \left\langle \nabla f_i^+, v \right\rangle & \text{if } v \text{ is inward to } \mathcal{D}_k^n(r)^+ \\ \left\langle \nabla f_i^-, v \right\rangle & \text{if } v \text{ is inward to } \mathcal{D}_k^n(r)^-. \end{cases}$$

Notice that the projections of ∇f_i^+ and ∇f_i^- onto $T_x\mathcal{S}$ coincide, so for $v \in T_x\mathcal{S}$ we have $D_v f_i = \left\langle \nabla f_i^+, v \right\rangle = \left\langle \nabla f_i^-, v \right\rangle$. As $\left\{\nabla f_i^+\right\}_{i=1}^n$ is an orthonormal basis, the lemma holds for $v \in T\mathcal{S}$ and hence also for v in a neighborhood U of $T\mathcal{S} \subset T\mathbb{D}_k^n(r)|_{\mathcal{S}}$. Since ∇f_0 is well defined on $\mathcal{S}$ and normal to $\mathcal{S}$, for any unit $v \in T\mathbb{D}_k^n(r)|_{\mathcal{S}} \setminus U$, we have $|D_v f_0| > 0$. The lemma follows from the compactness of the set of unit vectors in $T\mathbb{D}_k^n(r)|_{\mathcal{S}} \setminus U$. □

Notice that at p_k and $A(p_k)$ the gradients of f_k and f_0 are colinear. Using this we conclude

Addendum 4.9. *Let p_k be any of $p_1, \ldots p_n$. There is an $\varepsilon > 0$ so that for all $x \in B(p_k, \varepsilon) \cup B(A(p_k), \varepsilon)$ and all $v \in T_x\mathbb{D}_k^n(r)$, the index $j(v)$ in the previous lemma can be chosen to be different from k.*

Lemma 4.10. *There is a $\lambda > 0$ so that for all $v \in T_x\mathbb{D}_k^n(r)$ there is a $j(v) \in \{0, 1, \ldots, n\}$ so that*

$$|D_v f_z| > \lambda |v|$$

for all $z \in B(p_{j(v)}, d)$, provided d is sufficiently small.

Proof. If not then for each $i = 0, 1, \ldots, n$ there is a sequence $\{z_i^j\}_{j=1}^\infty \subset \mathbb{D}_k^n(r)$ with $\operatorname{dist}(z_i^j, p_i) < \frac{1}{j}$ and a sequence of unit $v^j \in T_{x^j}\mathbb{D}_k^n(r)$ so that

$$\left|D_{v^j} f_{z_i^j}\right| < \frac{1}{j}.$$

Choose the segments $x^j z_i^j$ and $x^j A\left(z_i^j\right)$ so that

$$\begin{aligned} \sphericalangle\left(\uparrow_{x^j}^{z_i^j},\ v^j\right) &= \sphericalangle\left(\Uparrow_{x^j}^{z_i^j},\ v^j\right) \text{ and} \\ \sphericalangle\left(\uparrow_{x^j}^{A\left(z_i^j\right)},\ v^j\right) &= \sphericalangle\left(\Uparrow_{x^j}^{A\left(z_i^j\right)},\ v^j\right). \end{aligned}$$

After passing to subsequences, we have $v^j \to v$, $x^j \to x$ and

$$\begin{aligned} x^j z_i^j &\to xp_i \\ x^j A\left(z_i^j\right) &\to xA\left(p_i\right), \end{aligned}$$

for some choice of segments xp_i and $xA\left(p_i\right)$. Using Lemma 3.8 and Corollary 3.11 we conclude

$$\begin{aligned} \left|\sphericalangle\left(\uparrow_{x^j}^{z_i^j}, v^j\right) - \sphericalangle\left(\uparrow_x^{p_i}, v\right)\right| &< \tau\left(\delta, \tau\left(\frac{1}{j}\middle|\operatorname{dist}\left(x, p_i\right)\right)\right), \\ \left|\sphericalangle\left(\uparrow_{x^j}^{A\left(z_i^j\right)}, v^j\right) - \sphericalangle\left(\uparrow_x^{A\left(p_i\right)}, v\right)\right| &< \tau\left(\delta, \tau\left(\frac{1}{j}\middle|\operatorname{dist}\left(x, A\left(p_i\right)\right)\right)\right). \end{aligned} \tag{4.10.1}$$

If $x \notin \mathcal{S}$, then the segments xp_i and $xA\left(p_i\right)$ are unambiguously defined, and so the previous inequality and the hypothesis $\left|D_{v^j} f_{z_i^j}\right| < \frac{1}{j}$, contradict the previous lemma and its addendum.

If $x \in \mathcal{S}$ and $v \in T_x\mathcal{S}$, then

$$\sphericalangle\left(\uparrow_x^{p_i}, v\right) \text{ and } \sphericalangle\left(\uparrow_x^{A\left(p_i\right)}, v\right)$$

are independent of the choice of the segments xp_i and $xA\left(p_i\right)$, so the hypothesis $\left|D_{v^j} f_{z_i^j}\right| < \frac{1}{j}$ together with the Inequalities 4.10.1 contradict the previous lemma and its addendum. Thus our result holds for $v \in T\mathcal{S}$ and hence also for v in a neighborhood U of $T\mathcal{S} \subset T\mathbb{D}_k^n(r)|_{\mathcal{S}}$.

For a unit vector $v \in T\mathbb{D}_k^n(r)|_{\mathcal{S}} \setminus U$, we saw in the proof of the previous lemma that for some $\lambda > 0$

$$\left|D_v f_0\right| > \lambda. \tag{4.10.2}$$

For $x \in \mathcal{S}$, we have unique segments xp_0 and $xA\left(p_0\right)$, so the hypothesis $\left|D_{v^j} f_{z_i^j}\right| < \frac{1}{j}$ and inequalities 4.10.1 contradict Inequality 4.10.2. □

Combining the proof of the previous lemma with Addendum 4.9, we get

Addendum 4.11. *Let p_k be any of $p_1, \dots p_n$. There is an $\varepsilon > 0$ so that for all $x \in B\left(p_k, \varepsilon\right) \cup B\left(A\left(p_k\right), \varepsilon\right)$ and all $v \in T_x\mathbb{D}_k^n(r)$, the index $j\left(v\right)$ in the previous lemma can be chosen to be different from k.*

Lemma 4.12. *There is a $\lambda > 0$ so that for all $v \in T\tilde{M}^\alpha$ there is a $j\left(v\right) \in \{0, 1, \dots, n\}$ so that*

$$D_v f_{j(v),d}^\alpha > \lambda\left|v\right|,$$

provided α is sufficiently large and d is sufficiently small.

Proof. If the lemma were false, then there would be a sequence of unit vectors $\{v^\alpha\}_{\alpha=1}^\infty$ with $v^\alpha \in T_{x^\alpha}\tilde{M}^\alpha$ such that for all i,

$$\left|D_{v^\alpha} f^\alpha_{i,d}\right| < \tau\left(\frac{1}{\alpha}, d\right).$$

Let $\lim_{\alpha\to\infty} x^\alpha = x \in \mathbb{D}^n_k(r)$. By Corollary 3.11, for any $\mu > 0$ there is a sequence $\{w^\alpha\}_{\alpha=1}^\infty$ with $w^\alpha \in \Sigma^\mu_{x^\alpha}$ such that

$$\sphericalangle\left(v^\alpha, w^\alpha\right) < \tau\left(\delta, \mu\right).$$

Since $\left|\nabla f^\alpha_{i,d}\right| \leq 2$,

$$\left|D_{w^\alpha} f^\alpha_{i,d}\right| < \tau\left(\delta, \mu, \frac{1}{\alpha}, d\right) \tag{4.12.1}$$

for all i. After passing to a subsequence, we conclude that $\left\{\gamma_{w^\alpha}|_{[0,\mu]}\right\}_{\alpha=1}^\infty$ converges to a segment $\gamma_w|_{[0,\mu]}$. By the previous lemma, there is a $\lambda > 0$ and a $j\left(w\right)$ so that for all $z \in B(p_{j(w)}, d)$,

$$\left|D_w f_z\right| > \lambda\left|w\right|, \tag{4.12.2}$$

provided d is small enough. Moreover, by Addendum 4.11 we may assume that

$$\begin{aligned} \operatorname{dist}\left(x, p_{j(w)}\right) &> 100d > \mu \text{ and} \\ \operatorname{dist}\left(x, A\left(p_{j(w)}\right)\right) &> 100d > \mu. \end{aligned} \tag{4.12.3}$$

By the Mean Value Theorem, there is a $z^\alpha_{j(w)} \in B\left(p^\alpha_{j(w)}, d\right)$ with

$$D_{w^\alpha} f^\alpha_{z^\alpha_{j(w)}} = D_{w^\alpha} f^\alpha_{j(w),d}. \tag{4.12.4}$$

Choose segments $x^\alpha z^\alpha_{j(w)}$ and $x^\alpha A(z^\alpha_{j(w)})$ in $\tilde{M}^\alpha$ so that

$$\begin{aligned} \sphericalangle\left(\uparrow^{z^\alpha_{j(w)}}_{x^\alpha}, w^\alpha\right) &= \sphericalangle\left(\Uparrow^{z^\alpha_{j(w)}}_{x^\alpha}, w^\alpha\right) \text{ and} \\ \sphericalangle\left(\uparrow^{A(z^\alpha_{j(w)})}_{x^\alpha}, w^\alpha\right) &= \sphericalangle\left(\Uparrow^{A(z^\alpha_{j(w)})}_{x^\alpha}, w^\alpha\right). \end{aligned}$$

After passing to a subsequence, we may assume that for some $z_{j(w)} \in B(p_{j(w)}, d)$, $x^\alpha z^\alpha_{j(w)}$ and $x^\alpha A(z^\alpha_{j(w)})$ converge to segments $xz_{j(w)}$ and $xA(z_{j(w)})$, respectively. By Lemma 3.8,

$$\begin{aligned} \left|\sphericalangle(\uparrow^{z^\alpha_{j(w)}}_{x^\alpha}, \gamma'_{w^\alpha}(0)) - \sphericalangle(\uparrow^{z_{j(w)}}_{x}, \gamma'_{w}(0))\right| &< \tau\left(\delta, \tau\left(1/\alpha|\mu, \operatorname{dist}\left(x, z_{j(w)}\right)\right)\right) \\ \left|\sphericalangle(\uparrow^{A(z^\alpha_{j(w)})}_{x^\alpha}, \gamma'_{w^\alpha}(0)) - \sphericalangle(\uparrow^{A\left(z_{j(w)}\right)}_{x}, \gamma'_{w}(0))\right| &< \tau\left(\delta, \tau\left(1/\alpha|\mu, \operatorname{dist}\left(x, A\left(z_{j(w)}\right)\right)\right)\right). \end{aligned}$$

Combining the previous two sets of displays with 4.12.3

$$\left|D_{w^\alpha} f^\alpha_{z^\alpha_{j(w)}} - D_w f_{z_{j(w)}}\right| < \tau\left(\delta, \tau\left(1/\alpha|\mu\right)\right). \tag{4.12.5}$$

So by Equation 4.12.4,

$$\left|D_{w^\alpha} f^\alpha_{j(w),d} - D_w f_{z_{j(w)}}\right| < \tau\left(\delta, \tau\left(1/\alpha|\mu\right)\right),$$

but this contradicts Inequalities 4.12.1 and 4.12.2. □

The first claim of Key Lemma 4.7 follows by combining the previous lemma with the fact that the differentials of the $\varphi^\eta_{x^\alpha}$'s are almost isometries.

Remark 4.13. *Note that when x^{α} is close to p_k or $A(p_k)$, the desired estimate*

$$\left|d(P_{\hat{\imath}_{x^\alpha}} \circ F)_{\varphi^{\eta}_{x^\alpha}(x^\alpha)}(v)\right| > \lambda |v|$$

holds with $P_{\hat{\imath}_{x^\alpha}} = P_{\hat{k}}$. This follows from Addendum 4.11 and the proof of the previous lemma.

Equicontinuity of Differentials. In this subsection, we establish the second part of the key lemma. If x^{α} is not close to one of the $p_k s$ or $A(p_k)\, s$ we will show the stronger estimate

$$\left|d(F)_{\varphi^{\eta}_{x^\alpha}(y)}(v) - d(F)_{\varphi^{\eta}_{x^\alpha}(x^\alpha)}(v)\right| < \frac{\lambda}{2}|v|. \tag{4.13.1}$$

So at such points, the second part of the key lemma holds with *any* choice of coordinate projection $P_{\hat{\imath}_{x^\alpha}}$.

For x^{α} close to p_k or $A(p_k)$, we will show

$$\left|d\left(P_{\hat{k}} \circ F\right)_{\varphi^{\eta}_{x^\alpha}(y)}(v) - d\left(P_{\hat{k}} \circ F\right)_{\varphi^{\eta}_{x^\alpha}(x^\alpha)}(v)\right| < \frac{\lambda}{2}|v|, \tag{4.13.2}$$

where λ is the constant whose existence was established in the previous section. Together with remark 4.13, this will establish the key lemma.

Suppose $B \subset \mathbb{D}^n_k(r)$ is (n,δ,r)–strained by $\{(a_i,b_i)\}_{i=1}^n$. Let $x,y \in B$ and let

$$\varphi^{\eta} : B \longrightarrow \mathbb{R}^n$$

be the map defined in 3.3.1 and [25]. Set

$$P_{x,y} := (d\varphi^{\eta})_y^{-1} \circ (d\varphi^{\eta})_x : T_x\mathbb{D}^n_k(r) \to T_y\mathbb{D}^n_k(r).$$

It follows that $P_{x,y}$ is a $\tau(\delta,\eta)$–isometry.

Lemma 4.14. *Let $B \subset \mathbb{D}^n_k(r)$ be (n,δ,r)–strained by $\{(a_i,b_i)\}_{i=1}^n$. Given $\varepsilon > 0$ and $x \in B$, there is a $\rho(x,\varepsilon) > 0$ so that the following holds.*

For all $k \in \{0,1,\ldots,n\}$, there is a subset $E_{k,x} \subset \{B(p_k,d) \cup B(A(p_k),d)\}$ with measure $\mu(E_{k,x}) < \varepsilon$ so that for all $z \in B(p_k,d) \setminus E_{k,x}$, all $y \in B(x,\rho(x,\varepsilon))$, and all $v \in \Sigma_x$,

$$\begin{aligned}\left|\sphericalangle(v,\uparrow_x^z) - \sphericalangle\left(P_{x,y}(v),\uparrow_y^z\right)\right| &< \tau(\varepsilon,\delta,\eta|\operatorname{dist}(x,z)) \text{ and}\\ \left|\sphericalangle\left(v,\uparrow_x^{A(z)}\right) - \sphericalangle\left(P_{x,y}(v),\uparrow_y^{A(z)}\right)\right| &< \tau(\varepsilon,\delta,\eta|\operatorname{dist}(x,A(z))).\end{aligned}$$

Proof. Let $C_x = \{z | z \in \operatorname{Cutlocus}(x) \text{ or } A(z) \in \operatorname{Cutlocus}(x)\}$ and set

$$E_{k,x} = B(C_x,\nu) \cap \{B(p_k,d) \cup B(A(p_k),d)\}.$$

Choose $\nu > 0$ so that $\mu(E_{k,x}) < \varepsilon$.

By Proposition 3.12, for each $z \in B(p_k,d) \setminus E_{k,x}$, there is a $\rho(x,z,\varepsilon)$ so that for all $y \in B(x,\rho(x,z,\varepsilon))$ and any choice of segment zy,

$$\operatorname{dist}(zx,zy) < \varepsilon,$$

where zx is the unique segment from z to x.

Making $\rho(x,z,\varepsilon)$ smaller and using Corollary 3.7, it follows that for any $\tilde{a}_i, \bar{a}_i \in B(a_i,\eta)$,

$$\begin{aligned}\left|\sphericalangle\left(\Uparrow_x^{\tilde{a}_i},\uparrow_x^z\right) - \sphericalangle\left(\Uparrow_y^{\bar{a}_i},\uparrow_y^z\right)\right| &< \tau(\delta,\varepsilon,\eta|\operatorname{dist}(x,z),\operatorname{dist}(y,z))\\ &= \tau(\delta,\varepsilon,\eta|\operatorname{dist}(x,z)).\end{aligned}$$

It follows that

$$\left|(d\varphi^{\eta})_x\left(\uparrow_x^z\right)-(d\varphi^{\eta})_y\left(\uparrow_y^z\right)\right|<\tau\left(\delta,\varepsilon,\eta|\mathrm{dist}\left(x,z\right)\right),$$

and hence

$$\sphericalangle\left(P_{x,y}\left(\uparrow_x^z\right),\uparrow_y^z\right)=\sphericalangle\left(\left(d\varphi^{\eta}\right)_y^{-1}\circ\left(d\varphi^{\eta}\right)_x\left(\uparrow_x^z\right),\left(\uparrow_y^z\right)\right)<\tau\left(\delta,\varepsilon,\eta|\mathrm{dist}\left(x,z\right)\right).$$

So for any $v\in\Sigma_x$,

$$\begin{aligned}\left|\sphericalangle\left(v,\uparrow_x^z\right)-\sphericalangle\left(P_{x,y}\left(v\right),\uparrow_y^z\right)\right| &\leq \left|\sphericalangle\left(v,\uparrow_x^z\right)-\sphericalangle\left(P_{x,y}\left(v\right),P_{x,y}\left(\uparrow_x^z\right)\right)\right|+\\ &\quad\left|\sphericalangle\left(P_{x,y}\left(v\right),P_{x,y}\left(\uparrow_x^z\right)\right)-\sphericalangle\left(P_{x,y}\left(v\right),\uparrow_y^z\right)\right|\\ &< \tau\left(\delta,\eta\right)+\tau\left(\varepsilon,\delta,\eta|\mathrm{dist}\left(x,z\right)\right)\\ &= \tau\left(\varepsilon,\delta,\eta|\mathrm{dist}\left(x,z\right)\right).\end{aligned}$$

Using Proposition 3.12 and the precompactness of $B\left(p_k,d\right)\setminus E_{k,x}$, we can then choose $\rho\left(x,z,\varepsilon\right)$ to be independent of $z\in B\left(p_k,d\right)\setminus E_{k,x}$. A similar argument gives the second inequality. □

Corollary 4.15. *Given any $\varepsilon>0$, there is a $\rho(\varepsilon)>0$ so that for any $x\in\mathbb{D}_k^n(r)$, $y\in B(x,\rho(\varepsilon))$, and $z\in B(p_i,d)\setminus E_{i,x}$, we have*

$$\left|D_v f_z-D_{P_{x,y}(v)}f_z\right|<\tau\left(\varepsilon,\delta,\eta|\mathrm{dist}\left(z,x\right),\mathrm{dist}\left(A\left(z\right),x\right)\right)$$

for all unit vectors $v\in\Sigma_x$.

Proof. Since $\mathbb{D}_k^n(r)$ is compact, the $\rho(\varepsilon,x)$ from the previous lemma can be chosen to be independent of x.

Given $x\in\mathbb{D}_k^n(r)$, $y\in B(x,\rho(\varepsilon))$, and $v\in\Sigma_x$, choose segments yz and $yA(z)$ so that

$$\begin{aligned}\sphericalangle\left(\uparrow_y^z,P_{x,y}\left(v\right)\right) &= \sphericalangle\left(\Uparrow_y^z,P_{x,y}\left(v\right)\right)\text{ and}\\ \sphericalangle\left(\uparrow_y^{A(z)},P_{x,y}\left(v\right)\right) &= \sphericalangle\left(\Uparrow_y^{A(z)},P_{x,y}\left(v\right)\right).\end{aligned}$$

Since the segments xz and $xA(z)$ are unique, the result follows from the formula for directional derivatives of distance functions, the previous lemma, and the chain rule. □

We can lift a strainer from $\mathbb{D}_k^n(r)$ to any $\tilde{M}^{\alpha}$ if $\mathrm{dist}_{GH}\left(\tilde{M}^{\alpha},\mathbb{D}_k^n(r)\right)$ is sufficiently small. So if x^{α} and y^{α} are sufficiently close, we define

$$P_{x^{\alpha},y^{\alpha}}:=\left(d\varphi^{\eta}\right)_{y^{\alpha}}^{-1}\circ\left(d\varphi^{\eta}\right)_{x^{\alpha}}:T_{x^{\alpha}}\tilde{M}^{\alpha}\to T_{y^{\alpha}}\tilde{M}^{\alpha}.$$

Lemma 4.16. *Let i be in $\{0,\ldots,n\}$. There is a $\rho>0$ so that for any $x^{\alpha}\in\tilde{M}^{\alpha}$, any $y^{\alpha}\in B(x^{\alpha},\rho)$, and any unit $v^{\alpha}\in T_{x^{\alpha}}\tilde{M}^{\alpha}$ we have*

$$\left|D_{v^{\alpha}}f_{i,d}^{\alpha}-D_{P_{x^{\alpha},y^{a}}(v^{\alpha})}f_{i,d}^{\alpha}\right|<\tau\left(\rho,\frac{1}{\alpha},\delta,\eta|\mathrm{dist}\left(x^{\alpha},p_i^{\alpha}\right),\mathrm{dist}\left(x^{\alpha},A\left(p_i^{\alpha}\right)\right)\right),$$

provided d is sufficiently small.

Proof. If not, then for any $\rho > 0$ and some $i = 0, 1, \ldots, n$, there would be a sequence of points $x^\alpha \to x \in \mathbb{D}^n_k(r)$, a sequence of unit vectors $\{v^\alpha\}_{\alpha=1}^\infty$ and a constant $C > 0$ that is independent of α, δ, and η so that

$$\begin{aligned} \left|D_{v^\alpha} f^\alpha_{i,d} - D_{P_{x^\alpha,y^a}(v^\alpha)} f^\alpha_{i,d}\right| &\geq C, \\ \mathrm{dist}\,(x, p_i) &\geq C, \text{ and} \\ \mathrm{dist}\,(x, A\,(p_i)) &\geq C \end{aligned} \tag{4.16.1}$$

for some $y^\alpha \in B(x^\alpha, \rho)$. Choose $\varepsilon > 0$ and take $\rho < \rho(\varepsilon)$ where $\rho(\varepsilon)$ is from the previous corollary. We assume $B\,(x, \rho(\varepsilon))$ is (n, δ, r)–strained. Let $y = \lim y^\alpha$ and $\mu > 0$ be sufficiently small. By corollary 3.11, there are sequences $\{w^\alpha\}_{\alpha=1}^\infty \in \Sigma^\mu_{x^\alpha}$ and $\{\tilde{w}^\alpha\}_{\alpha=1}^\infty \in \Sigma^\mu_{y^\alpha}$ so that

$$\begin{aligned} \sphericalangle\,(v^\alpha, w^\alpha) &< \tau\,(\delta, \mu) \\ \sphericalangle\,(P_{x^\alpha,y^a}\,(w^\alpha)\,, \tilde{w}^\alpha) &< \tau\,(\delta, \mu) \end{aligned} \tag{4.16.2}$$

and subsequences $\{\gamma_{w^\alpha}\}_{\alpha=1}^\infty$ and $\{\gamma_{\tilde{w}^\alpha}\}_{\alpha=1}^\infty$ converging to segments γ_w and $\gamma_{\tilde{w}}$ that are parameterized on $[0, \mu]$. Since $|\nabla f^\alpha_{i,d}| \leq 2$, we may assume for a possibly smaller constant C that

$$\left|D_{w^\alpha} f^\alpha_{i,d} - D_{\tilde{w}^\alpha} f^\alpha_{i,d}\right| \geq C.$$

Thus for some $z^\alpha \in B(p^\alpha_i, d)$ with $\mathrm{dist}_{\mathrm{Haus}}\,(z^\alpha, E_{i,x}) > 2\nu$,

$$|D_{w^\alpha} f^\alpha_{z^\alpha} - D_{\tilde{w}^\alpha} f^\alpha_{z^\alpha}| \geq \frac{C}{2}. \tag{4.16.3}$$

Passing to a subsequence, we have $z^\alpha \to z \in B(p_i, d) \setminus E_{i,x}$. As in the proof of Lemma 4.12 (Inequality 4.12.5), we have

$$\begin{aligned} |D_{w^\alpha} f^\alpha_{z^\alpha} - D_w f_z| &< \tau\,(\delta, \tau\,(1/\alpha|\mu)) \text{ and} \\ |D_{\tilde{w}^\alpha} f^\alpha_{z^\alpha} - D_{\tilde{w}} f_z| &< \tau\,(\delta, \tau\,(1/\alpha|\mu))\,. \end{aligned}$$

Thus,

$$\begin{aligned} |D_{w^\alpha} f^\alpha_{z^\alpha} - D_{\tilde{w}^\alpha} f^\alpha_{z^\alpha}| &\leq |D_{w^\alpha} f^\alpha_{z^\alpha} - D_w f_z| + |D_w f_z - D_{\tilde{w}} f_z| + |D_{\tilde{w}} f_z - D_{\tilde{w}^\alpha} f^\alpha_{z^\alpha}| \\ &< |D_w f_z - D_{\tilde{w}} f_z| + \tau\,(\delta, \tau\,(1/\alpha|\mu)) \\ &\leq \left|D_w f_z - D_{P_{x,y}(w)} f_z\right| + \left|D_{P_{x,y}(w)} f_z - D_{\tilde{w}} f_z\right| + \tau\,(\delta, \tau\,(1/\alpha|\mu)) \\ &\leq \tau\,(\varepsilon, \delta, \mu, \eta, \tau\,(1/\alpha|\mu)) \end{aligned}$$

by the previous corollary and Inequalities 4.16.1 and 4.16.2. Choosing $\varepsilon, \delta, \eta,$ μ, and $1/\alpha$ small enough, we have a contradiction to 4.16.3. □

The previous lemma, together with the definitions of Φ^α_d, $(\varphi^\eta)^{-1}$ and P_{x^α,y^a} establishes the estimates 4.13.1 and 4.13.2 and hence the second part of Key Lemma, completing the proof of Theorem 1.3, except in dimension 4.

5. Recognizing $\mathbb{R}P^4$

To prove Theorem 1.3 in dimension 4, we exploit the following corollary of the fact that $\mathrm{Diff}_+\left(S^3\right)$ is connected [3].

Corollary 5.1. *Let M be a smooth 4–manifold obtained by smoothly gluing a 4–disk to the boundary of the nontrivial 1–disk bundle over $\mathbb{R}P^3$. Then M is diffeomorphic to $\mathbb{R}P^4$.*

To see that our M^{α}s have this structure, we use standard triangle comparison and argue as we did in the part of Section 4 titled "Lower Bound on Differential" to conclude

Proposition 5.2. *For any fixed $\rho_0 > 0$, $f^{\alpha}_{0,d}$ does not have critical points on $M^{\alpha} \setminus \{B(p^{\alpha}_0, \rho_0) \cup B(A(p^{\alpha}_0), \rho_0)\}$, and $\nabla f^{\alpha}_{0,d}$ is gradient-like for* $\mathrm{dist}(A(p^{\alpha}_0), \cdot)$ *and $-\mathrm{dist}(p^{\alpha}_0, \cdot)$, provided α is sufficiently large and d is sufficiently small.*

Finally, using Swiss Cheese Volume Comparison (see 1.1 in [12]) we will show

Proposition 5.3. *There is a $\rho_0 > 0$ so that $\mathrm{dist}(p^{\alpha}_0, \cdot)$ does not have critical points in $B(p^{\alpha}_0, \rho_0)$, provided α is sufficiently large.*

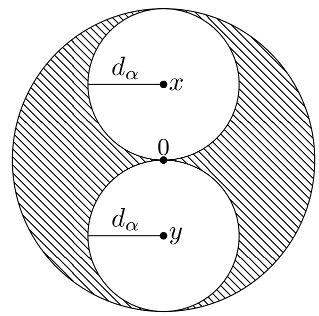

FIGURE 2. The model $\mathcal{D}^n_k(2d_\alpha)$.

Proof. Since $\mathrm{vol}\, M^{\alpha} \to \mathrm{vol}\, \mathcal{D}^n_k(r)$, $\mathrm{vol}\, B(p^{\alpha}_0, r) \to \mathrm{vol}\, \mathcal{D}^n_k(r)$. Via Swiss Cheese Volume Comparison (see 1.1 in [12]) we shall see that the presence of a critical point close to p^{α}_0 contradicts $\mathrm{vol}\, B(p^{\alpha}_0, r) \to \mathrm{vol}\, \mathcal{D}^n_k(r)$. Suppose q_α is critical for $\mathrm{dist}(p^{\alpha}_0, \cdot)$, and $\mathrm{dist}(p^{\alpha}_0, q_\alpha) = d_\alpha \to 0$. Let x, y be points in $\partial \mathcal{D}^n_k(d_\alpha)$ at maximal distance. By Swiss Cheese Comparison and 1.4 in [12],

$$\begin{aligned} \mathrm{vol}\, (B(q_\alpha, 2d_\alpha) \setminus B(p^{\alpha}_0, d_\alpha)) &\leq \mathrm{vol}\, (\mathcal{D}^n_k(2d_\alpha) \setminus \{B(x, d_\alpha) \cup B(y, d_\alpha)\}) \\ &= \mathrm{vol}\, (\mathcal{D}^n_k(2d_\alpha)) - 2\mathrm{vol}\, (\mathcal{D}^n_k(d_\alpha)). \end{aligned}$$

Since

$$\mathrm{vol}\, B(p^{\alpha}_0, d_\alpha) \leq \mathrm{vol}\, \mathcal{D}^n_k(d_\alpha),$$

we conclude

$$\begin{aligned} \mathrm{vol}\, (B(q_\alpha, 2d_\alpha)) &\leq \mathrm{vol}\, (\mathcal{D}^n_k(2d_\alpha)) - \mathrm{vol}\, (\mathcal{D}^n_k(d_\alpha)) \\ &< \kappa \cdot \mathrm{vol}\, \mathcal{D}^n_k(2d_\alpha) \end{aligned}$$

for some $\kappa \in (0,1)$. By relative volume comparison for $\rho \geq 2d_\alpha$,

$$\kappa > \frac{\mathrm{vol}\, B(q_\alpha, 2d_\alpha)}{\mathrm{vol}\, \mathcal{D}^n_k(2d_\alpha)} \geq \frac{\mathrm{vol}\, B(q_\alpha, \rho)}{\mathrm{vol}\, \mathcal{D}^n_k(\rho)}$$

or

$$\kappa \cdot \mathrm{vol}\, \mathcal{D}^n_k(\rho) > \mathrm{vol}\, B(q_\alpha, \rho).$$

Since

$$\begin{aligned} B(p^{\alpha}_0, r) &\subset B(q_\alpha, r + d_\alpha), \\ \mathrm{vol}\, B(p^{\alpha}_0, r) &< \kappa \cdot \mathrm{vol}\, \mathcal{D}^n_k(r + d_\alpha). \end{aligned}$$

Letting $d_\alpha \to 0$, we conclude that

$$\mathrm{vol}\, B(p^{\alpha}_0, r) < \kappa \cdot \mathrm{vol}\, \mathcal{D}^n_k(r),$$

a contradiction. $\square$

An identical argument shows

Proposition 5.4. *There is a* $\rho_0 > 0$ *so that* $\operatorname{dist}(A(p_0^\alpha), \cdot)$ *does not have critical points in* $B(A(p_0^\alpha), \rho)$, *provided* α *is sufficiently large.*

Combining the previous three propositions, we see that $(f_{0,d}^\alpha)^{-1}(0)$ is diffeomorphic to S^3. By Geometrization, $(f_{0,d}^\alpha)^{-1}(0) / \{\mathrm{id}, A\}$ is diffeomorphic to $\mathbb{R}P^3$. If ρ_0 is as in Proposition 5.2, it follows that $(f_{0,d}^\alpha)^{-1}([-\rho_0, \rho_0]) / \{\mathrm{id}, A\}$ is the nontrivial 1–disk bundle over $\mathbb{R}P^3$. $\tilde{M}^\alpha \setminus (f_{0,d}^\alpha)^{-1}([-\rho_0, \rho_0])$ consists of two smooth 4–disks that get interchanged by A. Thus M^α has the structure of Corollary 5.1 and is hence diffeomorphic to $\mathbb{R}P^4$.

Remark 5.5. *The proof of Perelman's Parameterized Stability Theorem* [16] *can substitute for Geometrization to allow us to conclude that* $f^{-1}(0) / \{\mathrm{id}, A\}$ *is homeomorphic and therefore diffeomorphic to* $\mathbb{R}P^3$. *The need to cite the proof rather than the theorem stems from the fact that the definition of admissible functions in* [16] *excludes* $f_{0,d}^\alpha$. *It is straightforward (but tedious) to see that the proof goes through for an abstract class that includes* $f_{0,d}^\alpha$.

The fact that $\mathbb{R}P^4$ *admits exotic differential structures can be seen by combining* [17] *with either* [4] *or* [5].

6. Purse Stability

We let Γ^n denote the group of twisted n–spheres. Recall that there is a filtration

$$\{e\} \subset \Gamma_{n-1}^n \subset \cdots \subset \Gamma_1^n = \Gamma^n$$

by subgroups, which are called Gromoll groups [9]. Rather than using the definition of the Γ_q^ns from [9], we use the equivalent notion from Theorem D in [15].

Definition 6.1. *Let*

$$f : S^{q-1} \times S^{n-q} \longrightarrow S^{q-1} \times S^{n-q}$$

be a diffeomorphism that satisfies

$$p_{q-1} \circ f = p_{q-1},$$

where

$$p_{q-1} : S^{q-1} \times S^{n-q} \longrightarrow S^{q-1}$$

is projection to the first factor. Then Γ_q^n *consists of those smooth manifolds that are diffeomorphic to*

$$D^q \times S^{n-q} \cup_f S^{q-1} \times D^{n-q+1}. \tag{6.1.1}$$

Theorem 6.2. *Let* $\{M^\alpha\}_{\alpha=1}^\infty$ *be a sequence of closed, Riemannian* n*–manifolds with*

$$\sec M^\alpha \geq k$$

so that

$$M_\alpha \longrightarrow P_{k,r}^n$$

in the Gromov-Hausdorff topology. Then for α *sufficiently large,* $M_\alpha \in \Gamma_{n-1}^n$.

Notice that a diffeomorphism $f : S^{n-2} \times S^1 \longrightarrow S^{n-2} \times S^1$ so that $p_{n-2} \circ f = p_{n-2}$ gives rise to an element of $\pi_{n-2}\left(\mathrm{Diff}_+\left(S^1\right)\right)$. If two such diffeomorphisms give the same homotopy class, then the construction 6.1.1 yields diffeomorphic manifolds (cf [15]). Since the group of orientation preserving diffeomorphisms of the circle deformation retracts to $SO(2)$, it follows that for $n \geq 4$, $\Gamma_{n-1}^n = \{e\}$. Since

$\Gamma^n = \{e\}$ for $n = 1, 2, 3$, we have $\Gamma^n_{n-1} = \{e\}$ for all n. Thus all but finitely many of the M^αs in Theorem 6.2 are diffeomorphic to S^n, and to prove Theorem 1.4 it suffices to prove Theorem 6.2.

The Model Submersion. Recall that we view $\mathcal{D}^n_k(r)$ as a metric r-ball centered at $p_0 = e_0$ in either $H^n_+ \subset \mathbb{R}^{1,n}$, $\{e_0\} \times \mathbb{R}^n \subset \mathbb{R}^{n+1}$, or $S^n \subset \mathbb{R}^{n+1}$, and we defined

$$p_i := \begin{cases} \cosh(r)e_0 + \sinh(r)e_i & \text{if } k = -1 \\ e_0 + re_i & \text{if } k = 0 \\ \cos(r)e_0 - \sin(r)e_i & \text{if } k = 1. \end{cases}$$

We let the totally geodesic hyperplane $H \subset \mathcal{D}^n_k(r)$ that defines $P^n_{k,r}$ be the one containing $p_0, p_1, \ldots, p_{n-1}$. We denote the singular subset of $P^n_{k,r}$ by $\mathcal{S}$, that is, $\mathcal{S}$ is the copy of S^{n-2} which is the boundary of the $(n-1)$–disk $\mathcal{D}^n_k(r) \cap H$. Thus $\{p_i\}^{n-1}_{i=1} \subset \mathcal{S}$.

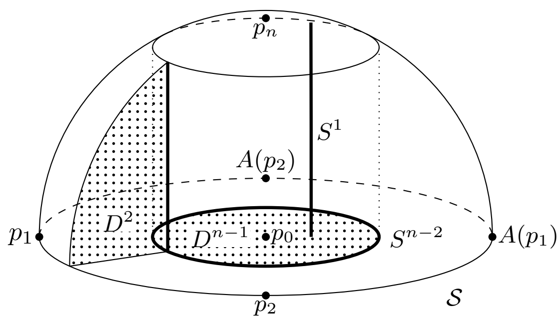

FIGURE 3. One side of $P^n_{k,r}$ for $n = 3$ and $k = 0$.

As the antipodal map $A : \mathcal{D}^n_k(r) \longrightarrow \mathcal{D}^n_k(r)$ commutes with the reflection R in H, it induces a well-defined involution of $P^n_{k,r}$, which we also call A. Note that $A : P^n_{k,r} \longrightarrow P^n_{k,r}$ restricts to the antipodal map of $\mathcal{S}$ and fixes the circle at maximal distance from $\mathcal{S}$.

For $i = 1, \ldots, n-1$, we view $\mathcal{S} \subset \mathcal{D}^n_k(r)$ and define f_i as in 4.1.1

$$f_i(x) := h_k \circ \operatorname{dist}(A(p_i), x) - h_k \circ \operatorname{dist}(p_i, x).$$

We let $\Psi : P^n_{k,r} \longrightarrow \mathbb{R}^{n-1}$ be defined by

$$\Psi = (f_1, f_2, \ldots, f_{n-1}).$$

Lifting The Model Submersion. Let $\{M^\alpha\}^\infty_{\alpha=1}$ be a sequence of closed, Riemannian n–manifolds with

$$\sec M^\alpha \geq k$$

so that

$$M_\alpha \longrightarrow P^n_{k,r}.$$

In contrast to the situation for the Crosscap, the isometry $A : P^n_{k,r} \longrightarrow P^n_{k,r}$ need not lift to an isometry of M^α. We nevertheless let $A : M^\alpha \longrightarrow M^\alpha$ denote any map that is Gromov-Hausdorff close to $A : P^n_{k,r} \longrightarrow P^n_{k,r}$.

As before, we define $f^\alpha_{i,d} : M^\alpha \longrightarrow \mathbb{R}$ by

$$f^\alpha_{i,d}(x) = \int_{z \in B\left(A\left(p^\alpha_i\right), d\right)} h_k \circ \operatorname{dist}(z, x) - \int_{z \in B\left(p^\alpha_i, d\right)} h_k \circ \operatorname{dist}(z, x). \tag{6.2.1}$$

We let $\Psi_d^\alpha : M^\alpha \longrightarrow \mathbb{R}^{n-1}$ be defined by

$$\Psi_d^\alpha = (f_{1,d}^\alpha, \ldots, f_{n-1,d}^\alpha).$$

The Handles. We identify $\mathbb{R}^{n-1}$ with

$$\mathbb{R}^{n-1} \equiv \operatorname{span}\{e_1, \ldots, e_{n-1}\} \subset \begin{cases} \mathbb{R}^{1,n} & \text{if } k = -1 \\ \mathbb{R}^{n+1} & \text{if } k = 0 \\ \mathbb{R}^{n+1} & \text{if } k = 1. \end{cases}$$

For small $\varepsilon > 0$, we set

$$\begin{aligned} E_0(\varepsilon) &:= (\Psi)^{-1}(D^{n-1}(0, r-\varepsilon)), \\ E_0^\alpha(\varepsilon) &:= (\Psi_d^\alpha)^{-1}(D^{n-1}(0, r-\varepsilon)), \\ E_1(\varepsilon) &:= (\Psi)^{-1}(\overline{A^{n-1}(0, r-\varepsilon, 2r)}), \text{ and} \\ E_1^\alpha(\varepsilon) &:= (\Psi_d^\alpha)^{-1}(\overline{A^{n-1}(0, r-\varepsilon, 2r)}), \end{aligned}$$

where $\overline{A^{n-1}(0, r-\varepsilon, 2r)}$ is the closed annulus in $\mathbb{R}^{n-1}$ centered at 0 with inner radius $r - \varepsilon$ and outer radius $2r$, and $D^{n-1}(0, r-\varepsilon)$ is the closed ball in $\mathbb{R}^{n-1}$ centered at 0 with radius $r - \varepsilon$.

Theorem 6.2 is a consequence of the next two lemmas.

Key Lemma 6.3. *For any sufficiently small $\varepsilon > 0$,*

$$\Psi_d^\alpha : E_0^\alpha(\varepsilon) \longrightarrow D^{n-1}(0, r-\varepsilon)$$

is a trivial S^1–bundle, provided α is sufficiently large and d is sufficiently small.

Let $\mathrm{pr} : \overline{A^{n-1}(0, r-\varepsilon, 2r)} \to \partial\left(D^{n-1}(0, r-\varepsilon)\right) = S^{n-2}$ be radial projection and set

$$\begin{aligned} g &:= \mathrm{pr} \circ \Psi : E_1(\varepsilon) \to \partial\left(D^{n-1}(0, r-\varepsilon)\right) \\ g_d^\alpha &:= \mathrm{pr} \circ \Psi_d^\alpha : E_1^\alpha(\varepsilon) \to \partial\left(D^{n-1}(0, r-\varepsilon)\right). \end{aligned}$$

Key Lemma 6.4. *There is an $\varepsilon > 0$ so that*

$$g_d^\alpha : E_1^\alpha(\varepsilon) \longrightarrow \partial\left(D^{n-1}(0, r-\varepsilon)\right)$$

is a trivial D^2–bundle over $\partial\left(D^{n-1}(0, r-\varepsilon)\right) = S^{n-2}$, provided α is sufficiently large and d is sufficiently small.

Since every space of directions of $P_{k,r}^n$ contains an isometrically embedded, totally geodesic copy of S^{n-3}, and every space of directions of $P_{k,r}^n \setminus \mathcal{S}$ contains an isometrically embedded, totally geodesic copy of S^{n-1}, we get the following. (Cf Proposition 3.2.)

Proposition 6.5. *There are $r, \delta > 0$ so that every point in the purse $P_{k,r}^n$ has a neighborhood B that is $(n-2, \delta, r)$–strained.*

For any neighborhood U of $\mathcal{S}$, there are $r, \delta > 0$ so that every point in $P_{k,r}^n \setminus U$ has a neighborhood B that is (n, δ, r)–strained.

Remark 6.6. *For $x \in \mathcal{S}$, the strainer $\{(a_i, b_i)\}_{i=1}^{n-2}$ can be chosen to lie in $\mathcal{S}$.*

Because the $f_i : P_{k,r}^n \longrightarrow \mathbb{R}$ are coordinate functions, $\Psi|_{\mathcal{D}_k^n(r) \cap H}$ differs from the identity by a translation. Using this and ideas from Section 4, we will be able to prove

Proposition 6.7. *There is a neighborhood U of $\mathcal{S} \subset P^n_{k,r}$ so that for any family of open sets $U^\alpha \subset M^\alpha$ with $U^\alpha \to U$, $g^\alpha_d|_{U^\alpha}$ is a submersion, provided α is sufficiently large and d is sufficiently small.*

We will show that our key lemmas hold for any $\varepsilon > 0$ so that

$$\Psi^{-1}\left(\overline{A^{n-1}(0, r-\varepsilon, r)}\right) \subset U.$$

Since $\{f_i\}_{i=1}^{n-1}$ are the $(n-1)$–coordinate functions for the standard embedding of $\mathcal{S} = S^{n-2} \subset \mathbb{R}^{n-1}$, we have

Lemma 6.8. *There is a $\lambda > 0$ so that for all $v \in T\mathcal{S}$, there is an j so that the j^{th}–component function of g satisfies*

$$|D_v(g_j)| > \lambda |v|.$$

As in Section 4, we have

Addendum 6.9. *Let p_k be any of $p_1, \ldots p_{n-1}$. There is an $\varepsilon > 0$ so that for all $x \in B(p_k, \varepsilon) \cup B(A(p_k), \varepsilon)$ and all $v \in T_x\mathcal{S}$, the index j in the previous lemma can be chosen to be different from k.*

To lift Lemma 6.8 to the M^αs, we need an analog of $T\mathcal{S}$ within each M^α, or better a notion of g^α_d–almost horizontal for each $U^\alpha \subset M^\alpha$. To achieve this, cover $\mathcal{S}$ by a finite number of $(n-2, \delta, r)$–strained neighborhoods $B \subset P^n_{k,r}$ with strainers $\{(a_i, b_i)\}_{i=1}^{n-2} \subset \mathcal{S}$. Let U be the union of this finite collection, and let $U^\alpha \subset M^\alpha$ converge to U.

Given $x^\alpha \in U^\alpha$, we now define a g^α_d–almost horizontal space at x^α as follows. Let B^α be a $(n-2, \delta, r)$–strained neighborhood for x^α with strainers $\{(a^\alpha_i, b^\alpha_i)\}_{i=1}^{n-2}$ that converge

$$\left(B^\alpha, \{(a^\alpha_i, b^\alpha_i)\}_{i=1}^{n-2}\right) \longrightarrow \left(B, \{(a_i, b_i)\}_{i=1}^{n-2}\right),$$

where $\left(B, \{(a_i, b_i)\}_{i=1}^{n-2}\right)$ is part of our finite collection of $(n-2, \delta, r)$–strained neighborhoods for points in $\mathcal{S} \subset P^n_{k,r}$. We set

$$H^{g^\alpha_d}_{x^\alpha} := \mathrm{span}_{i \in \{1, \ldots, n-2\}} \left\{\uparrow^{a^\alpha_i}_{x^\alpha}\right\},$$

where $\uparrow^{a^\alpha_i}_{x^\alpha}$ is the direction of *any* segment from x^α back to a^α_i. Regardless of this choice, $H^{g^\alpha_d}_{x^\alpha}$ satisfies the following Lemma, from which Proposition 6.7 follows.

Lemma 6.10. *There is a $\lambda > 0$ so that for all $x^\alpha \in U^\alpha$ and all $v \in H^{g^\alpha_d}_{x^\alpha}$, there is an j so that the j^{th}–component function of g^α_d satisfies*

$$\left|D_v\left((g^\alpha_d)_j\right)\right| > \lambda |v|,$$

provided U and d are sufficiently small and α is sufficiently large. In particular, $g^\alpha_d|_{U^\alpha}$ is a submersion.

Proof. Let $x_\alpha \to x$, and for all $j = 1, \ldots, n-1$, let $z^\alpha_j \to z_j \in B(p_j, d)$. If $x_\alpha z^\alpha_j$ converges to $x z_j$, then by Corollary 3.7,

$$\left|\sphericalangle\left(\uparrow^{a^\alpha_i}_{x^\alpha}, \uparrow^{z^\alpha_j}_{x^\alpha}\right) - \sphericalangle\left(\uparrow^{a_i}_{x}, \uparrow^{z_j}_{x}\right)\right| < \tau\left(\delta, 1/\alpha | \mathrm{dist}(x, z_j)\right).$$

Similarly for a sequence of segments $x_\alpha A\left(z_j^\alpha\right)$ converging to $xA\left(z_j\right)$, we have

$$\left|\sphericalangle\left(\uparrow_{x^\alpha}^{a_i^\alpha},\uparrow_{x^\alpha}^{A\left(z_j^\alpha\right)}\right)-\sphericalangle\left(\uparrow_{x}^{a_i},\uparrow_{x}^{A(z_j)}\right)\right|<\tau\left(\delta,1/\alpha|\mathrm{dist}\left(x,A\left(z_j\right)\right)\right).$$

Arguing as in the proof of Lemma 4.12, we have for all i and j,

$$\left|D_{\uparrow_{x^\alpha}^{a_i^\alpha}}\left(g_d^\alpha\right)_j-D_{\uparrow_x^{a_i}}\left(g\right)_j\right|<\tau\left(\delta,d,1/\alpha|\mathrm{dist}\left(x,p_j\right),\mathrm{dist}\left(x,A\left(p_j\right)\right)\right).$$

Since $v\in H_{x^\alpha}^{g_d^\alpha}=\mathrm{span}_{i\in\{1,\ldots,n-2\}}\left\{\uparrow_{x^\alpha}^{a_i^\alpha}\right\}$, the lemma follows from the previous display together with Lemma 6.8, Addendum 6.9, and the hypothesis that U is sufficiently small. □

Let $p_n\in\mathcal{D}_k^n\left(r\right)$ be as in 4.1.2, and let $Q:\mathcal{D}_k^n\left(r\right)\longrightarrow P_{k,r}^n$ be the quotient map. We abuse notation and call $Q\left(p_n\right)$, p_n. We define $f_n:P_{k,r}^n\rightarrow\mathbb{R}$ by

$$f_n(x):=h_k\circ\mathrm{dist}\left(\left(p_n\right),x\right)-h_k\circ\mathrm{dist}\left(p_0,x\right).$$

With a slight modification of the proof of Proposition 3.2, we get

Lemma 6.11. *There are $\delta,r>0$ so that for all $x\in E_0\left(\varepsilon/2\right)$ there is an (n,δ,r)–strainer $\left\{\left(a_i,b_i\right)\right\}_{i=1}^n$ with*

$$\left\{\left(a_i,b_i\right)\right\}_{i=1}^{n-1}\subset f_n^{-1}\left(l\right)$$

for some $l\in\mathbb{R}$.

We cover $E_0\left(\varepsilon/2\right)$ by a finite number of such (n,δ,r)–strained sets and make

Definition 6.12. *For $x\in E_0\left(\varepsilon/2\right)$, set*

$$H_x^\Psi:=\mathrm{span}_{i\in\{1,\ldots,n-1\}}\left\{\uparrow_x^{a_i}\right\},$$

where $\left\{\left(a_i,b_i\right)\right\}_{i=1}^{n-1}$ is as in the previous lemma.

Since $\Psi:E_0\left(\varepsilon/2\right)\longrightarrow D^{n-1}\left(r-\varepsilon/2\right)$ is simply orthogonal projection, we have

Lemma 6.13. *There is a $\lambda>0$ so that for all $x\in E_0\left(\varepsilon/2\right)$ and all $v\in H_x^\Psi$, there is an i so that*

$$\left|D_vf_i\right|>\lambda\left|v\right|.$$

To lift this lemma to the M^αs, we need a notion of Ψ_d^α–almost horizontal for each M^α. Given $z^\alpha\in E_0^\alpha\left(\varepsilon/2\right)$, we define a Ψ_d^α–almost horizontal space at z^α as follows. Let B^α be a (n,δ,r)–strained neighborhood for z^α with strainers $\left\{\left(a_i^\alpha,b_i^\alpha\right)\right\}_{i=1}^n$ that converge

$$\left(B^\alpha,\left\{\left(a_i^\alpha,b_i^\alpha\right)\right\}_{i=1}^n\right)\longrightarrow\left(B,\left\{\left(a_i,b_i\right)\right\}_{i=1}^n\right),$$

where $\left(B,\left\{\left(a_i,b_i\right)\right\}_{i=1}^n\right)$ is part of our finite collection of (n,δ,r)–strained neighborhoods for points in $E_0\left(\varepsilon/2\right)$ that comes from Lemma 6.11. We set

$$H_{z^\alpha}^{\Psi_d^\alpha}:=\mathrm{span}_{i\in\{1,\ldots,n-1\}}\left\{\uparrow_{z^\alpha}^{a_i^\alpha}\right\},$$

where $\uparrow_{z^\alpha}^{a_i^\alpha}$ is the direction of *any* segment from z^α back to a_i^α. Regardless of this choice, $H_{z^\alpha}^{\Psi_d^\alpha}$ satisfies the following Lemma, whose proof is nearly identical to the proof of Lemma 4.12.

Lemma 6.14. *There is a $\lambda > 0$ so that for all $z^\alpha \in E_0^\alpha(\varepsilon/2)$ and all $v \in H_{z^\alpha}^{\Psi_d^\alpha}$, there is an $i \in \{1, \ldots, n-1\}$ so that*

$$\left|D_v f_{i,d}^\alpha\right| > \lambda |v|,$$

provided α is sufficiently large and d is sufficiently small. In particular, $\Psi_d^\alpha|_{E_0^\alpha(\varepsilon/2)}$ is a submersion.

Proposition 6.15. *$E_1^\alpha(\varepsilon)$ is homeomorphic to $S^{n-2} \times D^2$, and $E_0^\alpha(\varepsilon)$ is homeomorphic to $D^{n-1} \times S^1$, provided α is sufficiently large and d is sufficiently small.*

Proof. First we show that $E_0^\alpha(\varepsilon)$ is connected. By the Stability Theorem [16], we have homeomorphisms $h_\alpha : P_k^n(r) \longrightarrow M^\alpha$ that are also Gromov–Hausdorff approximations (cf [10], [12] and [27]). Thus for α sufficiently large, we have

$$E_0^\alpha(\varepsilon) \subset h_\alpha(E_0(\varepsilon/2)).$$

Let $\rho^\alpha : M^\alpha \longrightarrow \mathbb{R}$ be defined by

$$\rho^\alpha(x) := |\Psi_d^\alpha(x)|.$$

Since $\Psi_d^\alpha|_{E_0^\alpha(\varepsilon/2)}$ is a submersion, it follows that ρ^α does not have critical points on $E_0^\alpha(\varepsilon/2) \setminus E_0^\alpha(2\varepsilon)$. By construction, the flow lines of $\nabla\rho^\alpha$ are transverse to the boundary of $E_0^\alpha(\varepsilon)$ and hence can be used to move $h_\alpha(E_0(\varepsilon/2))$ onto $E_0^\alpha(\varepsilon)$. It follows that $E_0^\alpha(\varepsilon)$ is connected.

Since $\Psi_d^\alpha|_{E_0^\alpha(\varepsilon)}$ is a proper submersion, it is a fiber bundle with contractible base $D^{n-1}(0, r-\varepsilon)$. Since the fiber is 1–dimensional and the total space is connected, we conclude that $E_0^\alpha(\varepsilon)$ is homeomorphic to $D^{n-1} \times S^1$.

We choose a homeomorphism $h_0 : E_0(\varepsilon/2) \longrightarrow E_0^\alpha(\varepsilon/2)$ so that

$$\begin{tikzcd}
E_0(\varepsilon/2) \arrow[rr, "h_0"] \arrow[dr, "\Psi_d"'] & & E_0^\alpha(\varepsilon/2) \arrow[dl, "\Psi_d^\alpha"] \\
& D^{n-1} &
\end{tikzcd}$$

commutes. Using the proof of the Gluing Theorem ([16], Theorem 4.6), we construct a homeomorphism $h : P_k^n(r) \longrightarrow M^\alpha$ so that

$$h = \begin{cases} h_0 & \text{on } E_0(\varepsilon) \\ h_\alpha & \text{on } E_1(\varepsilon/4). \end{cases}$$

It follows that $h(E_1(\varepsilon)) = E_1^\alpha(\varepsilon)$. Since $E_1(\varepsilon)$ is homeomorphic to $S^{n-2} \times D^2$, the result follows. $\square$

Proof of Key Lemma 6.4. By Proposition 6.7, $g_d^\alpha : E_1^\alpha(\varepsilon) \longrightarrow \partial D^{n-1}(0, r-\varepsilon) = S^{n-2}$ is a submersion. Since g_d^α is proper, g_d^α is a fiber bundle with two-dimensional fiber F. From the long exact homotopy sequence and Proposition 6.15, we conclude that F is a 2–disk. For $n \neq 4$, every D^2–bundle over S^{n-2} is trivial by Theorem 1 of [19]. When $n = 4$, $E_1^\alpha(\varepsilon)$ is a D^2–bundle over S^2 whose total space is homeomorphic to $S^2 \times D^2$. It follows for example from [33] that $E_1^\alpha(\varepsilon)$ is trivial in all cases, completing the proof of Key Lemma 6.4. $\square$

Proof of Key Lemma 6.3. Since $\Psi_d^\alpha|_{E_0^\alpha(\varepsilon)}$ is a proper submersion, $(E_0^\alpha(\varepsilon), \Psi_d^\alpha)$ is a fiber bundle over $D^{n-1}(0, r-\varepsilon)$ with one-dimensional fiber F. Since $E_0^\alpha(\varepsilon)$ is also homeomorphic to $D^{n-1} \times S^1$, it follows that the fiber is S^1. The base is contractible,

so the bundle is trivial. This completes the proof of Key Lemma 6.3 and hence the proofs of Theorems 6.2 and 1.4, establishing our Main Theorem. □

Double Disk Stability. The proof of Theorem 1.3 also yields

Corollary 6.16. *Let* $\{M_i\}_{i=1}^{\infty}$ *be a sequence of closed Riemannian* n*–manifolds with* $\sec M_i \geq k$ *so that*

$$M_i \longrightarrow \mathbb{D}_k^n(r)$$

in the Gromov-Hausdorff topology. Then all but finitely many of the M_i*s are diffeomorphic to* S^n*.*

Proof. In contrast to Theorem 4.1, we do not necessarily have an isometric involution of the M_is. Instead, we let $A : M_i \longrightarrow M_i$ be any map which is Gromov-Hausdorff close to $A : \mathbb{D}_k^n(r) \longrightarrow \mathbb{D}_k^n(r)$. We then define $f_{i,d}^{\alpha} : M_i \longrightarrow \mathbb{R}$ as in 6.2.1 and proceed as in the proof of Theorem 1.3. □

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